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Renormalization of QCD Coupling Constant in Terms of Physical Quantities

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A renormalization scheme is suggested where QCD input parameters—quark mass and coupling constant—are expressed in terms of gauge-invariant and infrared-stable quantities. For the QCD charge renormalization the quark anomalous electromagnetic moment is used. It turns out that in the renormalization procedure QED and QCD do not differ. The examination of the quark scattering amplitude indicates confinement phenomena in QCD.

Usually, in the charge and mass renormalization procedure the so-called MOM scheme is used, where the input parameters of the theory appearing in the action are expressed in terms of the on-mass-shell Green's functions (Itzykson and Zuber, 1980; Ramond 1989). Another widely accepted scheme, MS, deals only with the divergent parts of the Green's functions, not appealing to any condition on momenta. Evidently, due to renormalization invariance, any scheme is admissible.

In gauge theories like quantum electrodynamics (QED) and quantum chromodynamics (QCD) the input parameters are considered to be gauge invariant and infrared stable (i.e., not containing infrared divergences, generated by a massless gauge field). Let us for brevity call any quantity with these properties a physical quantity. Surely, the observables, being measurable, should be physical quantities, but the converse may not be true (as an example, consider any function of field strength $F_{\mu\nu}$ in QED). Since Green's functions are not infrared stable and depend on a gauge [in general their renormalization factors are nonlocal quantities (Basseto *et al.*, 1987)] the interpretation of the results obtained from the schemes mentioned above may be obscured.

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Renormalized physical quantities may share undesirable properties originating from the dynamics of nonphysical degrees of freedom, and the extraction of physical information may be complicated.

Therefore, from our point of view, in gauge theories the most illuminating approach would use a scheme operating only with physical degrees of freedom, thus allowing us to avoid such complications. The recipe is as follows: calculate as many physical quantities as there are input parameters and express the latter in terms of physical quantities. So, in QED as well as in QCD, containing two input parameters—coupling constant and fermion mass (for simplicity consider only one flavor)—we need two physical quantities:

$$\Sigma_\alpha = \Sigma_\alpha(g_0, m_0; n); \qquad \Sigma_\beta = \Sigma_\beta(g_0, m_0; n) \tag{1}$$

where g_0 and m_0 are input parameters, the dependence of Σ on momenta, being here irrelevant, is omitted, n is a space-time dimension, and hereafter we use dimensional regularization. In QED and QCD Σ_α and Σ_β have no limit at $n = 4$—regularization cannot be removed in (1). Now resolve g_0 and m_0 in terms of Σ_α, Σ_β, and n and substitute into the expression for any other physical quantity Σ_γ. In renormalizable theories, after all the calculations are performed, Σ_γ becomes finite in terms of Σ_α, Σ_β—regularization can be removed:

$$\begin{aligned}\lim_{n\to 4} \Sigma_\gamma(g_0, m_0; n) &= \lim_{n\to 4} \Sigma_\gamma(g_0(\Sigma_\alpha, \Sigma_\beta; n), m_0(\Sigma_\alpha, \Sigma_\beta; n); n)\\ &= \lim_{n\to 4} \Sigma^*_\gamma(\Sigma_\alpha, \Sigma_\beta; n)\\ &\equiv \sigma_\gamma < \infty\end{aligned} \tag{2}$$

This scheme, describing the renormalization procedure as the expression of physical quantities in terms of physical quantities,[3] allows us to avoid any significant difficulties and seems most transparent from a physical point of view.

The aim of the present paper is to develop such a scheme for QCD.

It is clear that we need two physical quantities. In (Tarrach (1981)) it was demonstrated that in the covariant gauge the solution of the equation

$$G^{-1}(p, g_0, m_0, \xi; n)\Psi_{\text{in}}(p) = 0 \tag{3}$$

in the framework of perturbation theory can be expressed as

$$m = m_0 + g_0^2\delta m_1(m_0; n) + g_0^4\delta m_2(m_0; n) + \cdots \tag{4}$$

where δm_1 and δm_2 do not depend on the gauge parameter ξ and are infrared

[3] A similar procedure (in a different context) was discussed already in the early days of QED (Dyson, 1949).

stable. In (3), G is a quark propagator and $\Psi_{\text{in}}(p)$ is the solution of the Dirac equation $(\gamma_\mu p^\mu - m)\Psi_{\text{in}}(p) = 0$. The investigation in an axial gauge (Japaridze *et al.*, 1991) confirms the gauge invariance and infrared stability of m. In (Japaridze *et al.* (1991)) it was shown that up to order g_0^6 the quark propagator has a simple pole at $p^2 = m^2$. So, the quark pole mass can be considered as one of the Σ's in relations (1), and to complete the scheme we have to find the second physical quantity in QCD.

We propose the quark anomalous electromagnetic moment, defined, as usual, from the amplitude of quark elastic scattering on an external electromagnetic field:

$$\langle p | j_\mu | p + k\rangle A^\mu(k) = \overline{\Psi}_{\text{in}}(p)\Big\{\gamma_\mu F_1(k^2) + \frac{i}{2}[\gamma_\mu, \gamma_\nu]k^\nu F_2(k^2)\Big\}\Psi_{\text{in}}(p + k)A^\mu(k) \tag{5}$$

The anomalous electromagnetic moment χ is defined as $F_2(0)$.

We calculate χ up to order g_0^6 regularizing all the divergences (ultraviolet and infrared) by means of space-time dimension n:

$$\chi = g_0^2\chi_1(m_0; n) + g_0^4\chi_2(m_0; n) \tag{6}$$

where χ_i is the i-loop contribution. The gauge-dependent terms cancel in χ_i; χ_1 is infrared stable and in χ_2 the infrared divergence appears. To obtain the expression of order g_0^4, containing only g_0-associated divergences, we must use the relation [see (4)]

$$m_0 = m - g_0^2\delta m_1(m_0, n) + O(g_0^4) = m - g_0^2\delta m_1(m, n) + O(g_0^4)$$

in (6), i.e., reexpand the loop expressions:

$$\begin{aligned}\chi &= g_0^2\chi_1(m - g_0^2\delta m_1(m; n); n) + g_0^4\chi_2(m; n)\\ &= g_0^2\chi_1(m; n) + g_0^4\left(\chi_2(m; n) - \delta m_1 \frac{\partial\chi_1(m; n)}{\partial m}\right)\end{aligned} \tag{7}$$

This reexpansion generates the infrared-divergent term $\partial\chi_1/\partial m$, which cancels the infrared divergence in χ_2.

So, χ is gauge invariant and infrared stable. Omitting the intermediate calculations, we quote the result:

$$\chi = C_F \frac{g_0^{*2}}{8\pi^2}\left\{1 - \frac{g_0^{*2}}{8\pi^2}\left[\frac{11C_A - 2n_f}{3}\left(\frac{1}{n-4} + \ln\frac{m^2}{4\pi\nu^2}\right) + \Phi + O(g_0^2, n-4)\right]\right\} \tag{8}$$

where $C_A \equiv N$, $C_F \equiv (N^2 - 1)/2N$ are the invariants of the $SU(N)$ group, $g_0^{*2} \equiv g_0^2 \nu^{n-4}$ is the dimensionless coupling constant, ν is a mass scale parameter, appearing in the framework of dimensional regularization (Itzykson and Zuber, 1980; Ramond, 1989), n_f is the number of flavors, m is the pole mass of the scattered quark, the external momenta obey $p^2 = (p + k)^2 = m^2$, and for the finite part Φ see the Appendix.

Thus in terms of the renormalization procedure QED and QCD do not differ—in both theories the input parameters can be expressed in terms of physical quantities. Surely, any scheme can be used, but our goal was to demonstrate that it is possible to renormalize the QCD coupling constant in terms of physical degrees of freedom. The scheme is described by the relations

$$m_0 = m_0(m, \chi; n), \qquad g_0 = g_0(m, \chi; n) \tag{9}$$

To use (9) we need the numerical values of m and χ. The gauge invariance and infrared stability of these quantities does not mean necessarily that they are measurable directly, but guarantees that their numerical values may be extracted from the experimental data. Of course, m and χ depend not only on m_0 and g_0, but also on all other input parameters of, say, the standard model, but for the considered problem it is enough to analyze only QCD corrections.

The scheme (9) may be not suitable from the point of view of numerical convergence—it depends on numerical values of m and χ. To improve he convergence, one has to introduce the so-called effective parameters (Itzykson and Zuber, 1980; Ramond, 1989) g_R and m_R, i.e., move to another scheme. It should be pointed out that some statements formulated in terms of effective parameters (say, the increasing of the QCD coupling constant in the infrared region, interpreted sometimes as a physical effect of increasing force between quarks at large distances) are scheme dependent and are not valid in another scheme. In other words, since the effective parameters are chosen arbitrarily, they cannot affect the physical results.

To see this, let us illustrate how the renormalization group equation and renormalization scheme arise in quantum field theory. It is transparent in dimensional regularization, where we have two parameters m_0 and $g_0^2 = g_0^{*2} v^{4-n}$. The mass scale v defines the dimension of g_0, providing the dimensionless action. The g_0^* and v are not independent:

$$g_0^2 = g_0^{*2}(v_1) v_1^{4-n} = g_0^{*2}(v_2) v_2^{4-n} \tag{10}$$

or

$$\frac{d g_0^{*2}(v)}{dv} + \frac{4 - n}{v} g_0^{*2}(v) = 0 \tag{11}$$

The renormalization group equation (11) can be presented in a familiar form by introducing the effective parameter $g_R(v)$ by means of the relation

$$g_0^{*2} = g_0^{*2}(g_R^2(v), v) \tag{12}$$

leading to

$$v \frac{dg_R^2(v)}{dv} = \beta(g_R^2, v) \tag{13}$$

where

$$\beta = \lim_{n \to 4} \frac{1}{\partial g_0^{*2}/\partial g_R^2} \left[(n-4) g_0^{*2} - \frac{\partial g_0^{*2}}{\partial v} \right] \tag{14}$$

The renormalization scheme is specified by relation (12); then from (14) we obtain the appropriate β-function. For example, the choice

$$g_0^{*2} = c_1(n) g_R^2(v) + c_2(n) g_R^4(v) + \cdots \tag{15}$$

results in

$$\beta(g_R) = \lim_{n \to 4} g_R^2(n-4) \left(1 - \frac{c_2(n)}{c_1(n)} g_R^2 + \cdots \right) \tag{16}$$

Particular c_i lead to particular schemes [e.g., MS is obtained if we choose $c_i = a_i/(n-4)^i$]. The introduction of $m_R(v)$ is based on the same argument.

So, the behavior of effective charge defined through any particular scheme (e.g., leading to asymptotic freedom), being scheme dependent, may not lead to any valuable results—the behavior and numerical value of effective parameters depend on our choice and are not defined from the theory alone.

Thus, the advantage of scheme (9), besides gauge invariance and infrared stability, is that it does not operate with effective parameters, allowing us to avoid conclusions that are insignificant from the physical point of view.

As becomes evident, the behavior of g_R and m_R does not lead to a scheme-independent conclusion, e.g., the absence of quarks and gluons in asymptotic states. On the other hand, from the renormalizability of QCD it follows that in the expansion of any physical quantity σ_γ (say, Wilson loop)

$$\sigma_\gamma = \sum_{i=0}^{\infty} \chi^i \sigma_{\gamma,i} \tag{17}$$

the coefficients $\sigma_{\gamma,i}$ are gauge invariant, infrared stable, and contain no uitarviolet divergences at $n = 4$, i.e., the $\sigma_{\gamma,i}$ are finite.

So, the following question arises: Does this lead us to conclusion that quark and gluon physical degrees of freedom are observable, i.e., do quarks and gluons appear in asymptotic states?

The answer may be obtained from the examination of the scattering matrix. Let us propose the following criterion: if at least one S-matrix element built up in terms of fields is finite, the appropriate quanta appear in asymptotic states as particles.

In QED it is well known that the electron elastic scattering amplitude is infrared divergent, and taking account of photon emission leads to cancellation of infrared singularities—only the inclusive cross sections are finite (Itzykson and Zuber, 1980; Ramond, 1989). That is why we may say that from Dirac–Maxwell equations there follows the existence of electrons and photons as observable particles. Of course, we *a priori* know that they exist and the S-matrix analysis is in accordance with the experimental data.

We consider the S-matrix element of quark scattering on an external electromagnetic field. Note first that the existence of $\chi \equiv F_2(0)$ [see (5)] does not mean at all that the amplitude of elastic scattering is finite—in full analogy with QED, the infrared divergences remain in the elastic amplitude after the mass and charge renormalization. Let us consider gluon emission, assuming that the inclusive cross section may be finite. According to the LSZ reduction technique (Itzykson and Zuber, 1980; Ramond, 1989), for a gluon with momentum q in an asymptotic state we have

$$\frac{\varepsilon_\rho(q)}{Z_3^{1/2}}\, q^2 D_{\mu\rho}(q) = \frac{\varepsilon_\mu(q)}{Z_3^{1/2}}\, q^2\, \frac{Z_3}{q^2} = Z_3^{1/2}\varepsilon_\mu(q) \tag{18}$$

where $\varepsilon_\mu(q)$ is the gluon polarization vector, $D_{\mu\rho}$ is the gluon propagator, and $Z_3^{1/2}$ is the gluon wave function renormalization factor. In order g_0^{*2} the contribution of the gauge field in the residue Z_3 is (Itzykson and Zuber, 1980; Ramond, 1989)

$$Z_3^A(q^2) = iC_A\, \frac{g_0^{*2} v^{4-n}}{2^{n+1}\pi^{n/2}}\, \frac{3n-2}{n-1}\, \frac{\Gamma^2(n/2-1)\Gamma(2-n/2)}{\Gamma(n-2)}\, (q^2)^{(n-4)/2} \tag{19}$$

where Γ is Eulers gamma function (Bateman and Erdelyi, 1973). Z_3 contributes to the QCD coupling constant renormalization. To obtain the amplitude we have to perform all the calculations before removing regularization: renormalize m_0 and g_0 by means of the relations (9), use [as we do for amplitude (5)] the conditions $p^2 = (p+k)^2 = m^2$, $q^2 = 0$, and then set $n = 4$. The condition $q^2 = 0$ means that $Z_3^A(0)$ can be equated to zero. This is analogous to the common procedure of vanishing of tadpole-type massless integrals—because of the analyticity of the theory in n we can find the region where the result is well defined and then analytically continue it in the desired region of n (Itzykson and Zuber, 1980; Ramond, 1989). So

$$Z_3(q^2) = Z_3^A(q^2) + Z_3^F(q^2) \underset{q^2 \to 0}{\to} Z_3^F(0) \tag{20}$$

i.e., only the fermionic field contribution survives in Z_3 at $q^2 \to 0$. This means that only part of the g_0 renormalization constant, namely the $C_A/2 - 2n_f/3$, is restored [compare with $(11C_A - 2n_f)/3$ in (8)]. In other words, $Z_3(0)$ does not renormalize g_0.

Therefore, if we consider the gluon emission to cancel the infrared divergences of the elastic amplitude, due to (20) the ultraviolet divergence arises and the inclusive cross section in order g_0^4 contains an unavoidable infinity. The S-matrix elements should be the same in any scheme, but the result is transparent in scheme (9), manipulating only the physical degrees of freedom and not using effective parameters.

Though the discussion above the S-matrix is not a proof, it can be considered as an indication of confinement phenomena already in the framework of perturbation theory. In other words, QCD may be an example of a field theory where fields do not necessarily refer directly to physical particles.[4]

The next step would be the consideration of the amplitude of colorless bound-state scattering. At the present time we have no definite result for this problem.

To conclude, in QCD it is possible to define the renormalization procedure operating only in the space of physical degrees of freedom. Though the use of relations (9) guarantees that the physical quantities built up in terms of quark and gluon fields are finite, this is not enough for the existence of these field quanta as physical particles. The examination of the quark scattering amplitude indicates that quark and gluon field quanta do not appear in asymptotic states.

APPENDIX

The finite part Φ is

$$\Phi = C_F\left(-\frac{55\pi^2}{9} - 8\gamma_E^2 - 3\zeta(3) + 2\pi^2 \ln 2 + \frac{493}{12}\right)$$
$$- \frac{2n_f\gamma_E}{3} + \frac{49n_f}{18} - 22 + \frac{26\pi^2}{9} + 4\gamma_E^2$$
$$+ C_A\left(\frac{131\pi^2}{18} + \frac{1675\gamma_E^2}{108} + \frac{17\gamma_E}{9} - \pi^2 \ln 2 + \frac{3}{2}\zeta(3) - \frac{959}{72}\right)$$

[4] A long time ago Schwinger (1962), starting from the idea that fields are more fundamental entities than particles, realized this in two-dimensional electrodynamics, where the particles, corresponding to fermionic degrees of freedom, are absent in the exact solution.

$$
\begin{aligned}
&- \frac{\pi^2}{2}\Delta_i^{1/2} - \Delta_i(6 + 4\ln\Delta_i) + \frac{5\pi^2}{2}\Delta_i^{3/2} \\
&+ \Delta_i^2\left[4\gamma_E^2 - \frac{2\pi^2}{3} - 3\ln^2\Delta_i + 4\ln\Delta_i - 8 + \left(\frac{4}{9}\ln\Delta_i - \frac{38}{27}\right)F_{i1}\right. \\
&\left. + \frac{4}{9}F'_{i1} + \left(\frac{4}{9}\ln\Delta_i - \frac{20}{27}\right)F_{i2} + \frac{4}{9}F'_{i2}\right] \\
&+ \Delta_i^3\left[\left(\frac{2}{3}\ln\Delta_i - \frac{11}{9}\right)F_{i3} + \frac{1}{3}F'_{i3} + (3 - 2\ln\Delta_i)F_{i4} - 2F'_{i4}\right] \qquad \text{(A1)}
\end{aligned}
$$

where $\Delta_i \equiv m_i^2/m^2$, $m_i \neq m$, the summation on $i = 1, 2, \ldots, n_f - 1$ is assumed,

$$
\begin{aligned}
F_{i1} &\equiv {}_3F_4\left(\left.\begin{matrix} 1, \frac{n+2}{2}, \frac{n-3}{2}, \frac{n-2}{2} \\ 2, n-2, \frac{n+1}{2} \end{matrix}\right| \Delta_i\right) \\
F_{i2} &\equiv {}_2F_3\left(\left.\begin{matrix} 1, \frac{n-1}{2}, \frac{n-2}{2} \\ \frac{n+1}{2}, \frac{n}{2} \end{matrix}\right| \Delta_i\right) \\
F_{i3} &\equiv {}_3F_4\left(\left.\begin{matrix} 1, \frac{n+2}{2}, \frac{n-1}{2}, \frac{n-2}{2} \\ \frac{n+1}{2}, 3, n-1 \end{matrix}\right| \Delta_i\right) \qquad \text{(A2)} \\
F_{i4} &\equiv {}_1F_2\left(\left.\begin{matrix} 1, \frac{n-2}{2} \\ \frac{n+2}{2} \end{matrix}\right| \Delta_i\right) \\
F'_{ij} &\equiv \frac{dF_{ij}}{dn}
\end{aligned}
$$

are generalized hypergeometric functions and their derivatives (Bateman and Erdelyi, 1973), considered at $n = 4$, $\gamma_E \approx 0.5771$ is Eulers constant, and

$$\zeta(3) \equiv \sum_{j=1}^{\infty} \frac{1}{j^3} \tag{A3}$$

is the Riemann zeta function.

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